

# Constrained optimal rearrangement problem leading to a new type obstacle problem

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## Abstract

We consider a new type of obstacle problem in the cylindrical domain  $\Omega = D \times (0, 1)$  arising from minimization of the functional

$$\int_{\Omega} \frac{1}{2} |\nabla u|^2 + \chi_{\{v>0\}} u dx,$$

where  $v(x') = \int_0^1 u(x', t) dt$ . We prove several existence and regularity results and show that the comparison principle does not hold for minimizers.

This problem is derived from a classical optimal rearrangement problem in a cylindrical domain, under the constraint that the force function does not depend on the  $x_n$  variable of the cylindrical axis.

## 1 Introduction

One of the classical problems in rearrangements theory is the minimization or maximization of the functional

$$\Phi(f) = \int_{\Omega} f u_f dx = \int_{\Omega} |\nabla u_f|^2 dx = \sup_{u \in W_0^{1,2}(\Omega)} \int_{\Omega} 2fu - |\nabla u|^2 dx, \quad (1)$$

where  $u_f$  is the unique solution of the Dirichlet boundary value problem

$$\begin{cases} -\Delta u_f(x) = f(x) & \text{in } \Omega, \\ u_f = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

and  $f$  belongs to the rearrangement class

$$\mathcal{R}(f_0) = \{f \in L^2(\Omega) \mid \mathcal{L}^n(\{f > \alpha\}) = \mathcal{L}^n(\{f_0 > \alpha\}) \text{ for all } \alpha \in \mathbb{R}\}$$

generated by a given function  $f_0 \in L^2(\Omega)$  (see [2], [3], [4], [7]).

The function  $f$  can be interpreted as an external force function and in certain applications, especially in 3D, it makes sense to consider force functions which

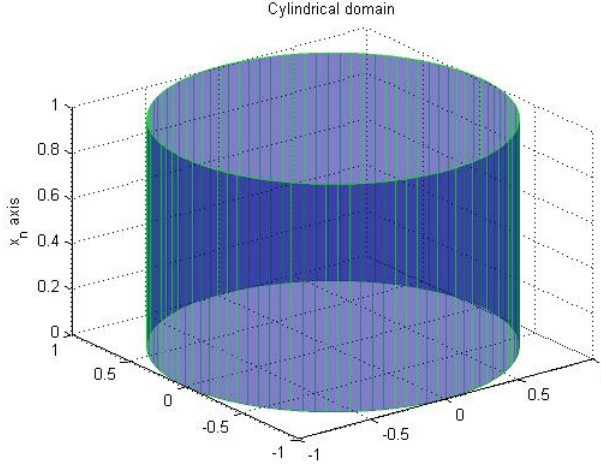
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are not dependent on one variable. In particular we will consider a barrel-like domain

$$\Omega = D \times (0, 1) \subset \mathbb{R}_{x'}^{n-1} \times \mathbb{R}_{x_n}.$$

and will restrict ourselves on functions  $f(x) = f(x')$ , which do not depend on the  $x_n$  variable. One can think of a stationary heating/cooling problem in a cylindrical container, where the function  $f$  does not depend on  $x_n$  variable (heating/cooling by vertical rods).



Without introducing new notations, in the sequel we will interpret functions from  $L^2(D)$  to be also defined as functions in  $L^2(\Omega)$  simply as  $f(x) = f(x')$ .

We only introduce the notation

$$\mathcal{R}_D(f_0) = \{f \in L^2(D) \mid \mathcal{L}^{n-1}(\{f > \alpha\}) = \mathcal{L}^n(\{f_0 > \alpha\}) \text{ for all } \alpha \in \mathbb{R}\} \subset \mathcal{R}(f_0)$$

indicating that the rearrangement class means to be defined in  $\Omega$  but consists only of functions which do not depend on  $x_n$  variable.

In this paper we will consider the minimization of

$$\Phi(f) = \int_{\Omega} f(x') u_f(x) dx = \int_D f(x') v_f(x') dx'$$

over the rearrangement class  $\mathcal{R}_D(\chi_{D_0})$ ,  $D_0 \subset D$ , where

$$v_f(x') = \int_0^1 u_f(x', x_n) dx_n, \quad (3)$$

as well as analyze its properties in Section 3. The main result of the paper is the introduction of the new type of obstacle problem naturally related to our rearrangement problem and some further regularity results (see Section 4).

## Acknowledgment

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## 2 Preliminaries

In this section we would like to present several mainly classical results.

**Lemma 2.1.** *Let*

$$-\Delta u = h(x) \text{ in } \Omega$$

*and  $|h(x)| \leq M$  is an integrable function in  $\Omega$ . Further assume  $\sup_{\Omega} |u| \leq N$ . Then*

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C(n, d)(M + N)$$

*where  $\Omega' \Subset \Omega$  and  $d = \text{dist}(\Omega', \Omega^c)$ .*

*Proof.* See Theorems 8.32, 8.34 in [8].  $\square$

**Lemma 2.2.** *Let  $\Omega$  be a domain with  $C^{1,\alpha}$  boundary and the functions  $u$  and  $h$  be as in Lemma 2.1. Further assume  $u = 0$  on  $\partial\Omega$ . Then*

$$\|u\|_{C^{1,\alpha}(\Omega)} \leq C(n, \partial\Omega)(M + N).$$

*Proof.* See Theorems 8.33, 8.34 in [8].  $\square$

**Lemma 2.3.** *Let  $\Omega = D \times (0, 1)$  and*

$$\begin{cases} -\Delta u(x) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

*and  $|h(x)| \leq M$  is an integrable function in. Further assume  $\sup_{\Omega} |u| \leq N$ . Then*

$$\|u\|_{C^{1,\alpha}(D' \times (0,1))} \leq C(n, d)(M + N),$$

*where  $d = \text{dist}(D', D^c)$ .*

*Moreover, if  $D$  has  $C^{1,\alpha}$  boundary then*

$$\|u\|_{C^{1,\alpha}(\Omega)} \leq C(n, \partial D)(M + N).$$

*Proof.* Let us extend the function  $u$  by the odd reflection into  $\tilde{\Omega} = D \times (-1, 1)$

$$\tilde{u}(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n \geq 0, \\ -u(x', -x_n) & \text{if } x_n < 0. \end{cases} \quad (5)$$

Let us check that  $-\Delta \tilde{u}(x) = \tilde{h}(x)$  weakly in  $D \times (-1, 1)$  where

$$\tilde{h}(x', x_n) = \begin{cases} h(x', x_n) & \text{if } x_n > 0, \\ -h(x', -x_n) & \text{if } x_n < 0 \end{cases} \quad (6)$$

is a bounded function.

$$\begin{aligned} \int_{\tilde{\Omega}} \nabla \tilde{u}(x) \nabla \phi(x) dx &= \int_{\tilde{\Omega}} \nabla \tilde{u}(x) \nabla (\phi(x) \varphi_{\delta}(x_n)) dx + \\ &\quad \int_{\tilde{\Omega}} \nabla \tilde{u}(x) \nabla (\phi(x) (1 - \varphi_{\delta}(x_n))) dx = I_1 + I_2 \end{aligned} \quad (7)$$

where

$$\varphi_\delta(t) = \begin{cases} 1 & \text{if } |t| < \delta/2, \\ 0 & \text{if } |t| > \delta \end{cases}$$

is an even function from  $C_0^\infty(\mathbb{R})$  with values in  $[0, 1]$ , such that  $|\varphi'(t)| \leq 4/\delta$ . Let us now estimate the integrals on the right hand side of (7).

$$\begin{aligned} I_1 &= \int_{\Omega} \nabla u \nabla [(\phi(x', x_n) - \phi(x', -x_n))\varphi_\delta(x_n)] dx = \\ &\quad \int_{\Omega} h(x) [(\phi(x', x_n) - \phi(x', -x_n))\varphi_\delta(x_n)] dx + \\ &\quad \underbrace{\int_{\partial\Omega} u(x) \partial_\nu [(\phi(x', x_n) - \phi(x', -x_n))\varphi_\delta(x_n)] d\sigma}_{=0} \rightarrow_{\delta \rightarrow 0} 0, \end{aligned} \quad (8)$$

where we have used the continuity of  $\phi \in C_0^\infty(\tilde{\Omega})$ . On the other hand

$$I_2 = \int_{\tilde{\Omega}} h(x) \phi(x', x_n) (1 - \varphi_\delta(x_n)) dx \rightarrow_{\delta \rightarrow 0} \int_{\tilde{\Omega}} h(x) \phi(x', x_n) dx. \quad (9)$$

The proof follows now from Lemmas 2.1 and 2.2.  $\square$

**Lemma 2.4.** *Let*

$$\begin{cases} -\Delta u(x) = f(x') & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (10)$$

*then*

$$u(x', x_n) = u(x', 1 - x_n). \quad (11)$$

*and the function  $v_f(x') = \int_0^1 u_f(x', x_n) dx_n$  satisfies the following equation*

$$\begin{cases} -\Delta_{x'} v = f(x') + 2\partial_\nu u(x', 0) & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases} \quad (12)$$

*Proof.* (11) follows from the uniqueness of the solution.

Let us take  $\phi_\delta(x) = \psi(x')\varphi_\delta(x_n)$ , where  $\psi \in C_0^\infty(D)$  and

$$\varphi_\delta(x_n) = \begin{cases} \frac{1}{\delta(1-\delta)} x_n & \text{if } x_n \in (0, \delta) \\ \frac{1}{1-\delta} & \text{if } x_n \in (\delta, 1-\delta) \\ \frac{1}{\delta(1-\delta)} - \frac{1}{\delta(1-\delta)} x_n & \text{if } x_n \in (1-\delta, 1). \end{cases}$$

$$\begin{aligned} \int_D f(x') \psi(x') dx' &= \int_{\Omega} f(x') \phi_\delta(x) dx = \int_{\Omega} \nabla u \nabla \phi_\delta dx \\ &= \int_{\Omega} \varphi_\delta(x_n) \nabla' u(x) \nabla' \psi(x') dx + \int_{\Omega} \psi(x') \partial_n u(x) \partial_n \varphi_\delta(x_n) dx. \end{aligned} \quad (13)$$

Passing to the limit as  $\delta \rightarrow 0$  we obtain

$$\int_{\Omega} \varphi_\delta(x_n) \nabla' u(x) \nabla' \psi(x') dx \rightarrow_{\delta \rightarrow 0} \int_{\Omega} \nabla' u(x) \nabla' \psi(x') dx = \int_D \nabla' v(x) \nabla' \psi(x') dx'$$

and using Lemma 2.3

$$\begin{aligned} \int_{\Omega} \psi(x') \partial_n u(x) \partial_n \varphi_{\delta}(x_n) dx = \\ \frac{1}{\delta(1-\delta)} \left[ \int_D \int_0^{\delta} \psi(x') \partial_n u(x) dx' dx_n - \int_D \int_{1-\delta}^1 \psi(x') \partial_n u(x) dx' dx_n \right] \rightarrow_{\delta \rightarrow 0} \\ \int_D \psi(x') [\partial_n u(x', 0) - \partial_n u(x', 1)] dx'. \end{aligned} \quad (14)$$

Thus

$$\int_D \nabla' v(x) \nabla' \psi(x') dx' = \int_D f(x') \psi(x') dx' - \int_D \psi(x') [\partial_n u(x', 0) - \partial_n u(x', 1)] dx'.$$

From (11) we obtain  $\partial_n u(x', 0) = -\partial_n u(x', 1)$ .  $\square$

**Lemma 2.5.** *Let  $0 \leq f_0 \leq 1$  and  $\bar{\mathcal{R}}(f_0)$  be the  $w^*$ -closure of  $\mathcal{R}(f_0)$  in  $L^2(D)$ . Then*

$$\bar{\mathcal{R}}(f_0) = \{h \mid 0 \leq h \leq 1, \text{ and } \int_D h dx' = \int_D f_0 dx'\}$$

*is convex and weakly compact in  $L^2$ . Moreover, the set of its extreme points is*

$$\text{ext}(\bar{\mathcal{R}}(f_0)) = \mathcal{R}(f_0).$$

*Proof.* See [2], [3], [5], [7].  $\square$

**Remark 2.1.** *We will use the notation  $\bar{\mathcal{R}}_D(f_0)$  indicating that the functions are considered in  $L^2(\Omega)$  but depend only on  $x' \in D$  variable.*

**Lemma 2.6.** *The functional  $\Phi$  (see (1)) is*

- (i) *weakly sequentially continuous in  $L^2$ ,*
- (ii) *strictly convex,*
- (iii) *Gâteaux differentiable. Moreover,  $\Phi'(f)$  can be identified with  $2u_f$  if we consider  $\Phi$  in  $L^2(\Omega)$  or  $2v_f$  if we consider  $\Phi$  in  $L^2(D)$ .*

*Proof.* The proof can be found in [4].  $\square$

**Lemma 2.7.** *For  $f, g \in L^2_+(D)$  there exists  $\tilde{f} \in \mathcal{R}(f)$  such that functional*

$$\int_D \tilde{f} g dx \leq \int_D h g dx,$$

*for all  $h \in \bar{\mathcal{R}}(f)$ .*

*Proof.* The proof can be found in [2].  $\square$

### 3 The constrained rearrangement problem

From now on we will assume that the generator function of the rearrangement class is a characteristic function  $f_0(x') = \chi_{D'}(x')$ , where  $D' \subset D$ . The functions  $u_f$  and  $v_f$  are defined in (2) and (3).

**Theorem 3.1.** *The relaxed minimization problem*

$$\min_{f \in \bar{\mathcal{R}}_D} \Phi(f)$$

has a unique solution  $\hat{f} \in \bar{\mathcal{R}}_D \setminus \mathcal{R}_D$ ,  $\hat{f} > 0$  in  $D$ , and there exists a constant  $\alpha > 0$  such that

$$\begin{aligned} \hat{v} = v_{\hat{f}} &\leq \alpha, \\ \{\hat{f} < 1\} &\subset \{\hat{v} = \alpha\} \\ \{\hat{v} < \alpha\} &\subset \{\hat{f} = 1\}. \end{aligned}$$

Moreover, the function  $\hat{U}(x) = \alpha - \hat{u}(x)$  is the minimizer of the convex functional

$$J(u) = \int_{\Omega} |\nabla U|^2 dx + 2 \int_D V^+ dx'. \quad (15)$$

among functions in  $W^{1,2}(\Omega)$  such that  $U - \alpha \in W_0^{1,2}(\Omega)$ .

*Proof.* By Lemmas 2.5 and 2.6

$$\min_{f \in \bar{\mathcal{R}}_D} \Phi(f)$$

has a solution since  $\bar{\mathcal{R}}_D$  is weakly compact and  $\Phi$  is weakly continuous. Further, the minimizer  $\hat{f} \in \bar{\mathcal{R}}_D$  is unique, since  $\Phi$  is strictly convex.

Let us now prove that  $\hat{f} \notin \mathcal{R}_D$ . The condition for the minimizer is

$$0 \in \partial\Phi(\hat{f}) + \partial\xi_{\bar{\mathcal{R}}_D}(\hat{f}),$$

where  $\partial\Phi$  is the sub-differential and

$$\xi_{\bar{\mathcal{R}}_D}(g) = \begin{cases} 0 & \text{if } g \in \bar{\mathcal{R}}_D, \\ \infty & \text{if } g \notin \bar{\mathcal{R}}_D, \end{cases}$$

see [6]. This means that  $-2\hat{v} \in \partial\xi_{\bar{\mathcal{R}}_D}(\hat{f})$ . Since

$$\partial\xi_{\bar{\mathcal{R}}_D}(\hat{f}) = \left\{ w \in L^2(D) : \xi_{\bar{\mathcal{R}}_D}(f) - \xi_{\bar{\mathcal{R}}_D}(\hat{f}) \geq \int_D (f - \hat{f})w dx' \right\}$$

we obtain

$$\int_D f \hat{v} dx' \geq \int_D \hat{f} \hat{v} dx'. \quad (16)$$

for all  $f \in \bar{\mathcal{R}}_D$ . By Lemma 2.7 there exists

$$\tilde{f} = \chi_{D_0} \in \text{ext}(\bar{\mathcal{R}}_D) = \mathcal{R}_D,$$

where  $D_0 \subset D$  and

$$\int_D \tilde{f} \hat{v} dx' = \int_D \hat{f} \hat{v} dx'. \quad (17)$$

**Claim 1:**

$$\alpha = \sup_{D_0} \hat{v} \leq \inf_{D \setminus D_0} \hat{v}. \quad (18)$$

This follows from (16) and (17). The idea of the proof is the following: if (18) fails to hold, we can rearrange the function  $\tilde{f}$  such that the integral  $\int_D \tilde{f} \hat{v} dx'$  decreases, by assigning the value 1 to  $\tilde{f}$  where  $\hat{v}$  is small and assigning the value 0 to  $\tilde{f}$  where  $\hat{v}$  is large (see [7], equation (3.17)).

**Claim 2:**

$$\hat{f} = \tilde{f} = \chi_{D_0} = 1, \text{ in } \{\hat{v} < \alpha\}. \quad (19)$$

The idea of the proof is the same as above: if (19) fails to hold then

$$\int_{D \setminus D_0} \hat{f} dx' = \int_{D_0} (1 - \hat{f}) dx' > 0,$$

thus we can replace the function  $\hat{f}$  by a function  $f \in \mathcal{R}_D$  which has larger values in  $\{\hat{v} < \alpha\} \subset D_0$  and smaller values in  $D \setminus D_0$ . As a result

$$\int_D f \hat{v} dx' < \int_D \hat{f} \hat{v} dx',$$

which contradicts (16).

**Claim 3:**

$$\{\hat{v} > \alpha\} \subset D^\# := \{\hat{f} = 0\}.$$

We know that  $\int_D \tilde{f} \hat{v} dx' = \int_D \hat{f} \hat{v} dx'$ , and

$$\begin{aligned} \int_D \tilde{f} \hat{v} dx' &= \int_{\{\hat{v} \geq \alpha\}} \tilde{f} \hat{v} dx' + \int_{\{\hat{v} < \alpha\}} \tilde{f} \hat{v} dx' = \\ &= \int_{\{\hat{v} \geq \alpha\}} \hat{f} \hat{v} dx' + \int_{\{\hat{v} < \alpha\}} \tilde{f} \hat{v} dx' = \int_D \hat{f} \hat{v} dx'. \end{aligned} \quad (20)$$

On the other hand  $\int_{\{\hat{v} < \alpha\}} \tilde{f} \hat{v} dx' = \int_{\{\hat{v} < \alpha\}} \hat{f} \hat{v} dx' = \int_{\{\hat{v} < \alpha\}} \hat{v} dx'$  and  $\tilde{f} = 0$  on  $\{\hat{v} > \alpha\}$ . This means that

$$\begin{aligned} \alpha \int_{\{\hat{v} \geq \alpha\}} \hat{f} dx' &= \alpha \int_{\{\hat{v} \geq \alpha\}} \tilde{f} dx' = \int_{\{\hat{v} \geq \alpha\}} \tilde{f} \hat{v} dx' = \\ &= \int_{\{\hat{v} \geq \alpha\}} \hat{f} \hat{v} dx' \geq \alpha \int_{\{\hat{v} \geq \alpha\}} \hat{f} dx', \end{aligned} \quad (21)$$

where the last inequality will be strict if  $\{\hat{v} > \alpha\} \cap \{\hat{f} > 0\}$  has a positive measure.

**Claim 4:**

$$D^\# \text{ has no interior. Thus } \hat{v} \leq \alpha.$$

From (12) and the Hopf's lemma it follows that

$$\Delta_{x'} \hat{v}(x') = -2\partial_\nu u(x', 0) > 0 \text{ in } \text{int}(D^\#)$$

and  $\hat{v} \geq \alpha$  in  $\text{int}(D^\#)$ . This means that there exists  $y \in \partial(\text{int}(D^\#))$  such that  $\hat{v}(y) = \beta > \alpha$ , which contradicts Claim 3 and continuity of  $\hat{v}$ .

**Claim 5:**

$$\hat{f} > 0.$$

We need to verify this only in  $\text{int}(\{\hat{v} = \alpha\})$  where

$$0 = \Delta_{x'} \hat{v} = -\hat{f}(x') - 2\partial_\nu \hat{u}(x', 0).$$

and the outer normal derivative of  $\hat{u}$  is not vanishing in  $D$  by Hopf lemma.

**Claim 6:**  $\hat{U} = \alpha - \hat{u}$  minimizes the functional (15).

From (1) we can obtain that  $\hat{U}$  minimizes the functional

$$I(U) = \int_{\Omega} |\nabla U|^2 + 2\hat{f}U dx = \int_{\Omega} |\nabla U|^2 dx + 2 \int_D \hat{f}V dx'$$

among  $U \in W^{1,2}(\Omega)$  such that  $U = \alpha$  on  $\partial\Omega$ . For any such function  $U$  we have

$$J(U) \geq I(U) \geq I(\hat{U}) = J(\hat{U}).$$

□

## 4 New type of obstacle problem

In this section we discuss the new type of obstacle problem, where the obstacle is acting not on the function  $u$ , but on the integral of  $u$  with respect to  $x_n$  variable. As a result, the free boundary is not a level set for the function  $u$ , which makes the analysis even more challenging.

### 4.1 Existence of solutions

**Theorem 4.1.** *Consider the minimization of the following convex functional*

$$J(u) = \int_{\Omega} |\nabla u|^2 dx + 2 \int_D v^+ dx'. \quad (22)$$

*among functions with prescribed boundary values  $u \in g + W_0^{1,2}(\Omega)$ , in a domain  $\Omega = D \times (0, 1)$ , where  $v(x') = \int_0^1 u(x', x_n) dx_n$ . We further assume that  $g$  is constant on  $D \times \{0\}$  and  $D \times \{1\}$  and that*

$$0 \leq g(x', x_n) \leq (1 - x_n)g(x', 0) + x_n g(x', 1) \quad (23)$$

*for all  $x' \in \partial D$ .*

*Then the functional  $J$  has a unique minimizer  $u$ , which satisfies the equation*

$$\Delta u(x) = \chi_{\{v>0\}} + \chi_{\{v=0\}} [\partial_\nu u(x', 0) + \partial_\nu u(x', 1)] \text{ in } \Omega. \quad (24)$$

**Remark 4.1.** *Here and later for a set  $A \subset D$  we define its characteristic function in  $\Omega$  as*

$$\chi_A(x) = \chi_A(x').$$



*Proof of Theorem 4.1.* Observe that

$$J(u) = \int_{\Omega} |\nabla u|^2 dx + 2 \int_D v^+ dx' = \int_{\Omega} |\nabla u|^2 + 2u\chi_{\{v>0\}} dx$$

and take the variations  $u_{\epsilon}(x) = u(x) + \epsilon\phi(x)$ , where  $\phi(x) \geq 0$ .

For  $\epsilon > 0$  the variation gives

$$2 \int_{\Omega} \nabla u \nabla \phi dx + 2 \int_{\Omega} \chi_{\{v \geq 0\}} \phi dx \geq 0$$

and for  $\epsilon < 0$

$$2 \int_{\Omega} \nabla u \nabla \phi dx + 2 \int_{\Omega} \chi_{\{v > 0\}} \phi dx \leq 0.$$

Thus

$$\int_{\Omega} \chi_{\{v > 0\}} \phi dx \leq - \int_{\Omega} \nabla u \nabla \phi dx \leq \int_{\Omega} \chi_{\{v \geq 0\}} \phi dx$$

and the distribution  $-\int_{\Omega} \nabla u \nabla \phi dx$  is a positive measure given by a function identified with  $\Delta u(x)$ , such that

$$- \int_{\Omega} \nabla u \nabla \phi dx = \int_{\Omega} \Delta u(x) \phi(x) dx$$

and

$$\chi_{\{v > 0\}} \leq \Delta u \leq \chi_{\{v \geq 0\}}. \quad (25)$$

**Claim 1:**  $\Delta u$  does not depend on  $x_n$ .

Let us consider the variation of the functional  $J$  with test function  $u_{\epsilon}(x) = u(x) + \epsilon\phi(x)$  where  $\phi(x) = \varphi(x', x_n) - \varphi(x', x_n - a)$  such that  $\varphi(x', x_n), \varphi(x', x_n - a) \in C_0^{\infty}(\Omega)$ . Then  $\int_0^1 \phi(x', x_n) dx_n = 0$  and thus the second term of the functional does not contribute to the variation. The contribution of the first term is

$$\int_{\Omega} \nabla u \nabla \varphi(x', x_n) dx - \int_{\Omega} \nabla u \nabla \varphi(x', x_n - a) dx = 0,$$

which proves that  $\Delta u$  does not depend on  $x_n$ .

**Claim 2:**

$$\Delta_{x'} v = \Delta u(x') - [\partial_{\nu} u(x', 0) + \partial_{\nu} u(x', 1)] \text{ in } D. \quad (26)$$

Follows from Lemma 2.4.

**Claim 3:**  $\{v < 0\} = \emptyset$ .

Assume  $\{v < 0\} = D^* \subset D$ . By continuity  $D^*$  is open,  $v = 0$  on  $\partial D^*$  and  $\Delta u = 0$  in  $D^* \times (0, 1)$ . By (26)

$$\Delta_{x'} v = -[\partial_{\nu} u(x', 0) + \partial_{\nu} u(x', 1)] \leq 0 \text{ in } D^*,$$

a contradiction. Here we have used the condition (23).

The equation (24) follows from (25) and (26).

□

## 4.2 Existence of weak second derivatives

In this section we apply the difference quotient argument to show the existence of weak second derivatives.

**Lemma 4.1.** *Let  $u$  be the minimizer of (22) in  $\Omega = D \times (0, 1)$  and  $u$  is constant on  $D \times \{0\}$  and on  $D \times \{1\}$ . Then for any compact  $\mathcal{C} \subset D$  there exists a constant  $C$  depending only on  $\text{dist}(\mathcal{C}, D^c)$  such that*

$$\int_{\mathcal{C} \times (0,1)} \left| \frac{\nabla(u(x+eh) - u(x))}{h} \right|^2 dx \leq C \int_{\Omega} \left| \frac{u(x+eh) - u(x)}{h} \right|^2 dx \quad (27)$$

for all  $|h| < \text{dist}(\mathcal{C}, D^c)/2$  and all directions  $e \perp e_n$ .

*Proof.* Let us take

$$\phi(x) = \psi(x')^2(u(x+eh) - u(x)),$$

where  $\psi \in C_0^\infty(D)$ ,  $0 \leq \psi \leq 1$ ,  $\psi(x') = 1$  for  $x' \in \mathcal{C}$ ,  $\psi(x') = 0$  for  $\text{dist}(x', \mathcal{C}) > \text{dist}(\mathcal{C}, D^c)/2$  and  $\nabla \psi \leq \frac{4}{\text{dist}(\mathcal{C}, D^c)}$ . Observe that the boundary values of the function

$$u(x) + t\phi(x) = t\psi(x')^2u(x+eh) + (1-t)\psi(x')^2u(x)$$

are the same as of  $u$ . Moreover, for  $t \in (0, 1)$

$$\int_0^1 u(x) + t\phi(x) dx_n = t\psi(x')^2v(x+eh) + (1-t)\psi(x')^2v(x) \geq 0$$

and we can consider the variations of the functional

$$I(u) = \int_{\Omega} |\nabla u|^2 + 2u dx. \quad (28)$$

instead of (22). From

$$J(u+t\phi) - J(u) = I(u+t\phi) - I(u) \geq 0$$

we obtain

$$0 \leq \int_{\Omega} \nabla u \nabla \phi + \phi dx$$

or

$$\int_{\Omega} \nabla u(x) \nabla (\psi(x')^2(u(x+eh) - u(x))) + (\psi(x')^2(u(x+eh) - u(x))) dx \geq 0. \quad (29)$$

Repeating the same argument as above for the function  $u(x+eh)$  in a slightly shifted domain and using the function  $u(x)$  for constructing a perturbation we can obtain the inequality

$$\int_{\Omega} \nabla u(x+eh) \nabla (\psi(x')^2(u(x) - u(x+eh))) + (\psi(x')^2(u(x) - u(x+eh))) dx \geq 0. \quad (30)$$

adding (29) and (30)

$$\begin{aligned}
0 &\geq \int_{\Omega} \nabla (u(x+eh) - u(x)) \nabla (\psi(x')^2 (u(x+eh) - u(x))) dx \\
&= \int_{\Omega} \psi(x')^2 |\nabla (u(x+eh) - u(x))|^2 dx + \\
&\quad \int_{\Omega} (u(x+eh) - u(x)) 2\psi(x') \nabla \psi \nabla (u(x+eh) - u(x)) dx \quad (31)
\end{aligned}$$

we arrive at

$$\begin{aligned}
&\int_{\Omega} \psi(x')^2 |\nabla (u(x+eh) - u(x))|^2 dx \leq \\
&\quad - \int_{\Omega} 2[(u(x+eh) - u(x)) \nabla \psi] \cdot [\psi(x') \nabla (u(x+eh) - u(x))] dx. \quad (32)
\end{aligned}$$

Now we use the inequality  $2|\mathbf{x} \cdot \mathbf{y}| \leq 2|\mathbf{x}|^2 + \frac{1}{2}|\mathbf{y}|^2$  to derive

$$\begin{aligned}
&-2[(u(x+eh) - u(x)) \nabla \psi] \cdot [\psi(x') \nabla (u(x+eh) - u(x))] \leq \\
&\quad 2|\nabla \psi|^2 |u(x+eh) - u(x)|^2 + \frac{1}{2} \psi(x')^2 |\nabla (u(x+eh) - u(x))|^2 \quad (33)
\end{aligned}$$

and obtain from (32)

$$\int_{\Omega} \psi(x')^2 |\nabla (u(x+eh) - u(x))|^2 dx \leq 4 \int_{\Omega} |\nabla \psi|^2 |u(x+eh) - u(x)|^2 dx \quad (34)$$

Taking  $C = \frac{64}{(\text{dist}(\mathcal{C}, D^c))^2}$  and dividing by  $h^2$  we obtain (27).  $\square$

**Lemma 4.2.** Let  $\Omega' \Subset \Omega$ ,  $\Omega'_\delta = \{x : \text{dist}(x, \Omega') < \delta\} \subset \Omega$ ,  $w \in L^2(\Omega)$  and

$$\int_{\Omega_\delta} \left| \frac{w(x + e_j h) - w(x)}{h} \right|^2 dx \leq C$$

for some constant  $C$  and all  $|h| < \delta$ .

Then the weak derivative  $\frac{\partial w}{\partial x_j}$  exists in  $\Omega'$  and

$$\int_{\Omega_\delta} \left| \frac{\partial w}{\partial x_j} \right|^2 dx \leq C.$$

*Proof.* See Lemma 7.24 in [8].  $\square$

**Lemma 4.3.** Assume  $\Omega' \Subset \Omega$  and  $u \in W^{1,2}(\Omega)$ . Then there exists a constant  $C > 0$  depending on dimension only such that

$$\int_{\Omega'} \left| \frac{u(x + e_j h) - u(x)}{h} \right|^2 dx \leq C \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^2 dx$$

for all  $|h| < \text{dist}(\Omega', \Omega^c)$ .

*Proof.* See Lemma 7.23 in [8].  $\square$

**Corollary 4.1.**

$$u \in W^{2,2}(D' \times (0, 1)), \text{ for any } D' \Subset D.$$

*Proof.* The existence  $\frac{\partial^2 u}{\partial x_i \partial x_j}$  in  $L^2(D' \times (\delta, 1 - \delta))$ , where  $1 \leq i \leq n - 1$  and  $1 \leq j \leq n$ , and integral bounds follow from Lemmas 4.1-4.3. The existence and integral bounds for  $\frac{\partial^2 u}{\partial x_n^2}$  follow from (25).

Now let us observe that because of constant boundary data on  $D \times \{0\}$  and  $D \times \{1\}$  we can extend the function to  $D \times (-1, 2)$  similarly as we have done it in Lemma 2.3. This is why we can let  $\delta = 0$ .  $\square$

**4.3 The comparison principle**

One of the interesting features of the functional  $J$  is that the comparison principle fails to hold for the minimizers  $u$  (see Remark 4.2 below). Here we prove that it holds for the functions  $v$  with constant boundary data.

**Theorem 4.2.** *Let  $u_1$  and  $u_2$  minimize (22) among functions with constant boundary data  $\alpha_1$  and  $\alpha_2$  respectively, and  $0 < \alpha_1 < \alpha_2$ . Then*

$$v_1(x') \leq v_2(x')$$

for  $x' \in D$ .

*Proof.* We prove the theorem in two steps.

**Step 1:** For  $\delta \geq 0$  let  $u_\delta$  be the minimizer of the convex functional

$$J_\delta(u) = \int_{\Omega} |\nabla u|^2 + \chi_{\{v > \delta\}} u dx$$

among the functions  $u \in W^{1,2}(\Omega)$  with boundary values  $u = \alpha_2$ . Let us prove that  $u_2 \leq u_\delta$ .

Assume  $\tilde{\Omega} = \{x \mid u_2(x) < u_\delta(x)\} \neq \emptyset$  and set  $u_3 = \min(u_2, u_\delta)$ . If  $\int_{\tilde{\Omega}} |\nabla u_2|^2 dx < \int_{\tilde{\Omega}} |\nabla u_\delta|^2 dx$  then

$$J_\delta(u_3) < J_\delta(u_\delta). \quad (35)$$

Otherwise if  $\int_{\tilde{\Omega}} |\nabla u_2|^2 dx \geq \int_{\tilde{\Omega}} |\nabla u_\delta|^2 dx$  then

$$J(u_3) < J(u_2). \quad (36)$$

Equations (35) and (36) contradict the fact that  $u_\delta$  and  $u_2$  are minimizers.

**Step 2:** For  $\delta = \alpha_2 - \alpha_1$  we have  $u_\delta = u_1 + \delta$ , where  $u_\delta$  is as in Step 1. From Step 1,

$$\partial_\nu u_1 \leq \partial_\nu u_2 \text{ on } D \times \{0\} \text{ and } D \times \{1\}$$

On the other hand by (12)

$$\Delta_{x'} v_1 = \chi_{\{v_1 > 0\}} [1 - \partial_\nu u_1(x', 0) - \partial_\nu u_1(x', 1)] \text{ in } D \quad (37)$$

and

$$\Delta_{x'} v_2 = \chi_{\{v_2 > 0\}} [1 - \partial_\nu u_2(x', 0) - \partial_\nu u_2(x', 1)] \text{ in } D. \quad (38)$$

Since

$$[1 - \partial_\nu u_1(x', 0) - \partial_\nu u_1(x', 1)] \geq [1 - \partial_\nu u_2(x', 0) - \partial_\nu u_2(x', 1)]$$

we can use the comparison principle for the classical obstacle problem to obtain  $v_1 \leq v_2$ .  $\square$

**Remark 4.2.** The comparison principle does **not** hold for the functions  $u_1$  and  $u_2$  in Theorem 4.2. Particularly, in the set  $\{v_2 = 0\} \subset \{v_1 = 0\}$ , where

$$\int_0^1 u_1(x', x_n) dx_n = \int_0^1 u_2(x', x_n) dx_n = 0 \quad (39)$$

but  $u_1 \not\equiv u_2$ .

*Proof.* Assume the comparison principle does hold and  $u_1 \leq u_2$ . Then from (39) it follows that  $u_1 \equiv u_2$  in  $\{v_2 = 0\} \times (0, 1)$ . Let us now consider the function  $w = u_2 - u_1 \geq 0$ . By (24)

$$\Delta w = \begin{cases} 0 & \text{in } (\{v_1 > 0\} \cup \{v_2 = 0\}) \times (0, 1), \\ 1 - 2\partial_\nu u_1 & \text{in } (\{v_1 = 0\} \setminus \{v_2 > 0\}) \times (0, 1), \end{cases} \quad (40)$$

and by (25)  $\Delta w \geq 0$ . Since the function  $w$  is positive at the boundary and vanishes in the set where  $u_1 = u_2$  is not constant and thus, by Hopf lemma,  $\partial_n w > 0$  in  $\{v_2 = 0\} \times \{0\}$ . This contradicts the fact of  $u_1 \equiv u_2$  on  $\{v_2 = 0\} \times (0, 1)$ .  $\square$

#### 4.4 Remarks on free boundary regularity

Let  $u$  and  $v$  be like in Theorem 4.1. From Lemma 2.4 and equation (24) it follows that the function  $v$  is the solution of the following obstacle problem

$$\Delta v = \chi_{\{v > 0\}} h(x'), \quad (41)$$

where

$$0 \leq h(x') = 1 - \partial_\nu u(x', 0) - \partial_\nu u(x', 1) \in C^\alpha(D).$$

In the points of the free boundary  $x' \in \partial\{v > 0\} \cap D$ , where  $h(x') > 0$  we can apply the Theorem 7.2 in [1] and obtain that

either  $x'$  is a regular point and the free boundary is  $C^{1,\alpha}$  smooth,

or  $x'$  is a singular point, i.e.  $\lim_{r \rightarrow 0} \frac{|\{v=0\} \cap B_r(x')|}{|B_r(x')|} = 0$ , and the free boundary in the ball  $B_r(x')$  has a minimum diameter less than  $\sigma(r)$ , for some given modulus of continuity  $\sigma$ .

Observe that it is possible to have solutions of (41) with singular free boundary point  $x'$ ,  $h(x') = 0$ , such that the free boundary in a ball  $B_r(x')$  is of order  $r$ .

To authors best knowledge for the minimizers of (22) it is not excluded that there could exist a singular free boundary point  $x'$ , with

$$1 - \partial_\nu u(x', 0) - \partial_\nu u(x', 1) = 0,$$

where the minimal diameter of the free boundary in a ball  $B_r(x')$  is of order  $r$ . This analysis is subject of ongoing research.

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